Causal relation between regions I and IV of the Kruskal extension

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Summary

By extending the exterior Schwarzschild spacetime in two opposite directions with the Kruskal method, we get an extension which has the same $T-X$ spacetime diagram as has the conventional Kruskal extension, while allowing its regions I and IV to correspond to different directions of the original spacetime. We further extend the exterior Schwarzschild spacetime in all directions and get a 4-dimensional form of the Kruskal extension. The new form of extension includes the conventional one as a part of itself. From the point of view of the 4-dimensional form, region IV of the conventional extension does not belong to another universe but is a portion of the same exterior Schwarzschild spacetime that contains region I. The two regions are causally related: particles can move from one to the other.

It is known that the conventional Kruskal extension contains four parts of spacetime, a black hole, a white hole, and two regions of the exterior Schwarzschild spacetime [1]. The two regions were believed to belong to different universes, and messages can not be exchanged between them. In the following we show that this interpretation is not correct.

The standard metric form of the exterior Schwarzschild spacetime is [2]

$$ds^2 = -(1-\frac{2M}{r})dt^2 + \frac{1}{1-\frac{2M}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad (r > 2M). \tag{1}$$

Let

$$\left\{ \begin{array}{c} t = t \\ x = r\sin\theta\cos\phi \\ y = r\sin\theta\sin\phi \\ z = r\cos\theta \end{array} \right. \qquad (r > 2M). \tag{2}$$

Then (1) can be written as

$$
\begin{aligned}
ds^2 = &-(1-\tfrac{2M}{r})dt^2 + [\tfrac{2Mx^2}{r^2(r-2M)}+1]dx^2 + [\tfrac{2My^2}{r^2(r-2M)}+1]dy^2 + [\tfrac{2Mz^2}{r^2(r-2M)}+1]dz^2 \\
&+\tfrac{4M}{r^2(r-2M)}(xydxdy+yzdydz+zxdzdx) \qquad\qquad (r>2M),
\end{aligned} \tag{3}
$$

where

$$
r=\sqrt{x^2+y^2+z^2}. \tag{4}
$$

Now we consider the extension of the exterior Schwarzschild spacetime in two opposite directions.

We notice from the transformation (2) that when $\theta=\pi/2$ and $\phi=0,\pi$ then

$$
y=z=0 \qquad and \qquad x=\pm r \qquad (r>2M), \tag{5}
$$

where $x=+r$ corresponds to $\phi=0$ and $x=-r$ corresponds to $\phi=\pi$. That is, while $x>0$ corresponds to one direction, $x<0$ corresponds to the opposite one. In this situation, from (3) and (4) one has

$$
ds^2=-(1-\frac{2M}{|x|})dt^2+\frac{|x|}{|x|-2M}dx^2 \qquad \forall(t,x,y,z)\in\widetilde{x}_+\cup\widetilde{x}_-, \tag{6}
$$

where $\widetilde{x}_+\equiv\{(t,x,y,z)|\infty>t>-\infty,\infty>x>2M,y=0,z=0\}$ is a portion of the exterior Schwarzschild spacetime in one direction and $\widetilde{x}_-\equiv\{(t,x,y,z)|\infty>t>-\infty,-2M>x>-\infty,y=0,z=0\}$ is another portion of the spacetime in the opposite direction. It can be verified that the transformation,

$$
\left\{\begin{array}{c}
T=(\frac{|x|}{2M}-1)^{1/2}e^{|x|/4M}sh\frac{t}{4M}\\
X=(\frac{|x|}{2M}-1)^{1/2}e^{|x|/4M}ch\frac{t}{4M}\\
y=y\\
z=z
\end{array}\right. \qquad \forall(t,x,y,z)\in\widetilde{x}_+ \tag{7}
$$

and

$$
\left\{\begin{array}{c}
T=(\frac{|x|}{2M}-1)^{1/2}e^{|x|/4M}sh\frac{t}{4M}\\
X=-(\frac{|x|}{2M}-1)^{1/2}e^{|x|/4M}ch\frac{t}{4M}\\
y=y\\
z=z
\end{array}\right. \qquad \forall(t,x,y,z)\in\widetilde{x}_-, \tag{8}
$$

will rewrite (6) in the form

$$
ds^2=\frac{32M^3e^{-|x|/2M}}{|x|}(-dT^2+dX^2) \qquad \forall(T,X,y,z)\in\widetilde{X}_+\cup\widetilde{X}_-, \tag{9}
$$

where T and X satisfy

$$
X^2-T^2=(\frac{|x|}{2M}-1)e^{|x|/2M} \qquad (|x|>2M), \tag{10}
$$

and $\widetilde{X}_+ \equiv \{(T,X,y,z)|\infty > T > -\infty, \infty > X > 0, y = 0, z = 0; X^2 - T^2 > 0\}$, $\widetilde{X}_- \equiv \{(T,X,y,z)|\infty > T > -\infty, 0 > X > -\infty, y = 0, z = 0; X^2 - T^2 > 0\}$.

The above transformation, denoted f_x, is a 1-1 map,with $f_x(\widetilde{x}_+) = \widetilde{X}_+$, $f_x(\widetilde{x}_-) = \widetilde{X}_-$ and $f_x(\widetilde{x}_+ \cup \widetilde{x}_-) = \widetilde{X}_+ \cup \widetilde{X}_-$. Note that, if relation (7) is applied in the transformation for both $\widetilde{x}_+$ and $\widetilde{x}_-$, then the map will no longer be 1-1 (even though the metric form of (9) is maintained).

Keeping the metric form of (9) but allowing $X^2 - T^2 > -1$ instead of $X^2 - T^2 > 0$ (these correspond to $|x| > 0$ and $|x| > 2M$ respectively), we have an extension of the part, $\widetilde{x}_+ \cup \widetilde{x}_-$, of an exterior Schwarzschild spacetime, confined in two opposite directions, in a manifold $\widetilde{X}$, $\widetilde{X} \equiv \{(T,X,y,z)|\infty > T > -\infty, \infty > X > -\infty, y = 0, z = 0\}$ which is an R^2 space. The new extension is 2-dimensional and has the same $T - X$ spacetime diagram as has the conventional Kruskal extension. (Note that, here $|x| = r$ according to (4) and (5).) However, for the new extension, while region I of the diagram (i.e. $\widetilde{X}_+$) corresponds to a portion of the exterior Schwarzschild spacetime in one direction, $\widetilde{x}_+$, region IV (i.e. $\widetilde{X}_-$) corresponds to none other than a portion in the opposite direction, $\widetilde{x}_-$.

This result led us further to consider the extension of the exterior Schwarzschild spacetime in an equatorial plane.

In the spacetime where

$$\theta = \frac{\pi}{2}, \tag{11}$$

(1) becomes

$$ds^2 = -(1 - \frac{2M}{r})dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2 d\phi^2 \qquad (r > 2M, \theta = \frac{\pi}{2}). \tag{12}$$

The following transformation

$$\begin{cases} T = (\frac{r}{2M} - 1)^{1/2} e^{r/4M} sh\frac{t}{4M} \\ R = (\frac{r}{2M} - 1)^{1/2} e^{r/4M} ch\frac{t}{4M} \\ \theta = \theta \\ \phi = \phi \end{cases} \qquad \forall (t,r,\theta,\phi) \in \widetilde{r}_{\pi/2}, \tag{13}$$

where $\widetilde{r}_{\pi/2} \equiv \{(t,r,\theta,\phi)|\infty > t > -\infty, \infty > r > 2M, \theta = \frac{\pi}{2}, 2\pi \geq \phi \geq 0\}$, rewrites (12) in the form

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dT^2 + dR^2) + r^2 d\phi^2 \qquad \forall (T,R,\theta,\phi) \in \widetilde{R}_{\pi/2}, \tag{14}$$

where T, R and r are related by

$$R^2 - T^2 = (\frac{r}{2M} - 1)e^{r/2M} \qquad (r > 2M), \tag{15}$$

and $\widetilde{R}_{\pi/2} \equiv \{(T, R, \theta, \phi) | \infty > T > -\infty, \infty > R > 0, \theta = \frac{\pi}{2}, 2\pi \geq \phi \geq 0; R^2 - T^2 > 0\}$.

The above transformation, denoted f_r, is a 1-1 map, with $f_r(\widetilde{r}_{\pi/2}) = \widetilde{R}_{\pi/2}$, where $\widetilde{r}_{\pi/2}$ is an R^3 space with an infinite length hollow column and $\widetilde{R}_{\pi/2}$ is an R^3 with two head-to-head opposite hollow cones extending to infinity.

In the same way, the extension of $\widetilde{R}_{\pi/2}$ is realized by keeping the metric form of (14) but allowing $R^2 - T^2 > -1$ instead of $R^2 - T^2 > 0$ (they correspond to $r > 0$ and $r > 2M$ respectively). In this extension we find that a black hole is inside one of the two cones and a white hole is inside the other cone. This extension is 3-dimensional and can be realized by rotating the conventional Kruskal extension around its T axis.

With the same method, the whole exterior Schwarzschild spacetime (in all directions) can be extended by making the following transformation

$$\begin{cases} T = (\frac{r}{2M} - 1)^{1/2} e^{r/4M} sh \frac{t}{4M} \\ R = (\frac{r}{2M} - 1)^{1/2} e^{r/4M} ch \frac{t}{4M} \\ \theta = \theta \\ \phi = \phi \end{cases} \qquad \forall (t, r, \theta, \phi) \in \widetilde{r}, \tag{16}$$

where $\widetilde{r} \equiv \{(t, r, \theta, \phi) | \infty > t > -\infty, \infty > r > 2M, \pi \geq \theta \geq 0, 2\pi \geq \phi \geq 0\}$, which rewrites (1) in the form

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad \forall (T, R, \theta, \phi) \in \widetilde{R}, \tag{17}$$

where

$$R^2 - T^2 = (\frac{r}{2M} - 1)e^{r/2M} \qquad (r > 2M), \tag{18}$$

and $\widetilde{R} \equiv \{(T, R, \theta, \phi) | \infty > T > -\infty, \infty > R > 0, \pi \geq \theta \geq 0, 2\pi \geq \phi \geq 0; R^2 - T^2 > 0\}$. Then, keeping the metric form of (17), we extend the range of $R^2 - T^2 > 0$ to $R^2 - T^2 > -1$ (they correspond to $r > 2M$ and $r > 0$ respectively). This yields

$$ds^2 = \frac{32M^3 e^{-r/2M}}{r}(-dT^2 + dR^2) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \qquad \forall (T, R, \theta, \phi) \in \widetilde{K}, \tag{19}$$

where

$$R^2 - T^2 = (\frac{r}{2M} - 1)e^{r/2M} \qquad (r > 0), \tag{20}$$

and $\widetilde{K} \equiv \{(T, R, \theta, \phi) | \infty > T > -\infty, \infty > R > 0, \pi \geq \theta \geq 0, 2\pi \geq \phi \geq 0; R^2 - T^2 > -1\}$.

This extension is 4-dimensional, and contains the extension of the equatorial plane as a 3-dimensional section, and the conventional extension as a 2-dimensional section. From the point of view of the 4-dimensional extension, region IV of the conventional extension does not belong

to a different universe but is a portion of the same exterior Schwarzschild spacetime that contains region I. Particles can move from one to the other. For example, the world line of a particle in a circular orbit outside the Schwarzschild black hole (with $r =$ constant $> 2M$) can be a line around the T axis and towards the direction of T from region I (where $\theta = \frac{\pi}{2}, \phi = 0$) through the regions of ($\theta = \frac{\pi}{2}, 0 < \phi < \pi$) to region IV (where $\theta = \frac{\pi}{2}, \phi = \pi$), shown in the spacetime diagram of the 3-dimensional head-to-head double-cone extension.

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